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Torsion of an Elastic Non-homogeneous Layer by Two Circular Discs

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Abstract. Two circular discs of different radii on the opposite faces of an infinite, non-homogeneous elastic layer, whose rigidity is assumed to vary with two cylindrical coordinates r, z by a power law ($\mu = \mu_{\alpha,\beta} r^{\alpha} z^{\beta}$), are forced to rotate through two different angles of rotation. The rest of each surface is kept stress free. Using the Hankel integral transform, this problem is shown to lead to two pairs of dual integral equations, the solution of which is governed by two simultaneous Fredholm integral equations of the second kind. The latter may be solved either numerically or by iteration (in the case of sufficiently large values of the layer's thickness compared to the maximum of the radii of the circles and for $\beta = 0$). The solutions for some particular cases previously investigated are recovered by assigning specific numerical values to physical and geometrical parameters.

Expressions for some quantities of physical importance, such as the torques applied on the two surfaces and the stress intensity factors, are obtained for the two surfaces and some numerical values are displayed graphically.

1. Introduction

The problem pertaining to the determination of the stress and displacement in a semi-infinite isotropic, homogeneous, elastic solid $z \geq 0$ when a circular area ($0 \leq r \leq a$) of the boundary surface $z = 0$ is forced to rotate through an angle about its axis, the rest of the surface being kept stress free, was first considered by Reissner and Sagoci [1] and Sagoci [2]. Since then, important contributions have been achieved by many authors such as Collins [3] and Sneddon [4, 5]. Rostovtsev [6], Bycroft [7], Ufliand [8], Gladwell [9] and Hassan [10] solved the Reissner–Sagoci problem for an elastic homogeneous layer of finite thickness when the lower face is either stress free or rigidly clamped.

The problem for a non-homogeneous, semi-infinite solid, a case of geological importance, was considered by Protsenko [11,12], Kassir [13], Chuaprasert and Kassir [14], Singh [15] and Selvaduria *et al.* [16]; for a non-homogeneous large thick plate with rigidity varying with the depth by Protsenko [17] and Hassan [18]; for a non-homogeneous large thick plate with rigidity varying with the coordinates r, z by a power law ($\mu = \mu_{\alpha,\beta} z^{\alpha}/r^{\beta}$) by Hassan [19].

In refs. [8] and [17], the second boundary surface of the layer is considered to be rigidly clamped, but in refs. [10], [18] and [19] it may be either stress free or rigidly clamped.

Singh and Dhaliwal [20] studied a homogeneous layer ($0 \leq z \leq h$) of finite thickness h on which two circular discs ($z = 0, 0 \leq r \leq a$ and $z = h, 0 \leq r \leq b$) are forced to rotate through different angles.

In the present paper we consider the generalization of [20] to the case when the modulus of rigidity is governed by the power law $\mu = \mu_{\alpha,\beta} r^{\alpha} z^{\beta} (\alpha \geq 0, 0 \leq \beta < 1)$, $\mu_{\alpha,\beta}$ being a constant. Two discs on the two surfaces ($z = 0, z = h$) with two different radii (a, b) are forced to rotate through two different angles of rotation (γ, δ), respectively, the rest of each surface being kept stress free. The solution of this problem is reduced to a pair of simultaneous

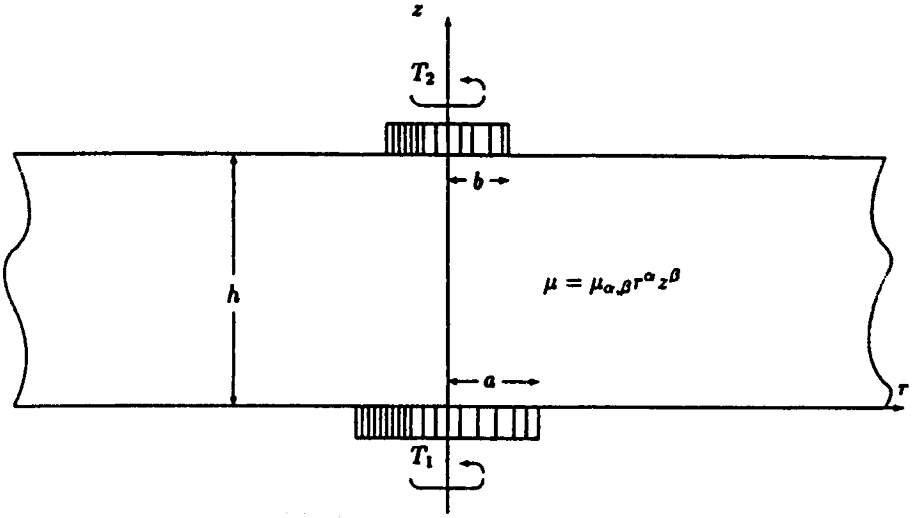

Figure 1. Geometry of the problem.

Fredholm integral equations in two auxiliary functions. This pair is solved numerically for the special case $\beta = 0$.

Solutions for some special cases have been derived and are shown to agree with the existing solutions.

Expressions are given for some quantities of physical importance, such as the torques applied at the two surfaces ($z = 0, z = h$) and the stress intensity factors at the rims ($z = 0, r = a$ and $z = h, r = b$).

The numerical results obtained clearly show that for fixed h as α increases, both the torque on the surface $z = 0$ and the stress intensity factor tend to their corresponding values for the half-space.

Curves are presented for the torque acting on the surface $z = 0$ against the relative thickness h/a for different values of γ, δ, α, and b/a.

2. Formulation of the Problem

We consider an infinitely large, thick, isotropic, non-homogeneous elastic layer bounded by the planes ($z = 0, z = h$) of a cylindrical coordinate system (r, θ, z) (Figure 1). Let the rigid discs of radii a and b be attached to the faces ($z = 0, z = h$) respectively such that the line joining the centers of the discs coincides with the z-axis.

The shear modulus of the solid is taken in the form

$$\mu = \mu_{\alpha,\beta} r^\alpha z^\beta, \tag{1}$$

where $\alpha \geq 0, 0 \leq \beta < 1$ and $\mu_{\alpha,\beta}$ is a constant.

The discs are forced to rotate through different angles γ and δ, respectively, the rest of the faces $z = 0, h$ being stress free.

For the axisymmetrical torsion problem, the only nonvanishing displacement component is the circumferential one, u_θ. The nonvanishing stress components $\sigma_{\theta_r}, \sigma_{\theta_z}$ are related to u_θ through the relations

$$\sigma_{\theta_r} = \mu r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right), \qquad \sigma_{\theta_z} = \mu \frac{\partial u_\theta}{\partial z}. \tag{2}$$

The only nontrivially satisfied equilibrium equation is

$$\frac{\partial \sigma_{\theta_r}}{\partial r}+\frac{\partial \sigma_{\theta_z}}{\partial z}+\frac{2\sigma_{\theta_r}}{r}=0. \tag{3}$$

Substituting (2) in (3) and taking (1) into account, we obtain the following partial differential equation for the displacement component u_θ:

$$\frac{\partial^2 u_\theta}{\partial r^2}+(1+\alpha)\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{\beta}{z}\frac{\partial u_\theta}{\partial z}+\frac{\partial^2 u_\theta}{\partial z^2}=0. \tag{4}$$

The boundary conditions are

$$\begin{aligned}
u_\theta(r,0) &= \gamma r, && (0\le r\le a), && (5)\\
\sigma_{\theta_z}(r,0) &= 0, && (r>a), && (6)\\
u_\theta(r,h) &= \delta r, && (0\le r\le b), && (7)\\
\sigma_{\theta_z}(r,h) &= 0, && (r>b), && (8)\\
\lim_{r\to\infty} u_\theta(r,z) &= 0. && && (9)
\end{aligned}$$

3. Reduction to a Pair of Integral Equations

The solution of Equation (4) which satisfies Condition (9) takes the form

$$u_\theta(r,z)=r^{1-\nu}z^p\int_0^\infty \lambda^p J_\nu(\lambda r)[A(\lambda)I_p(\lambda z)+B(\lambda)I_{-p}(\lambda z)]\,\mathrm{d}\lambda, \tag{10}$$

where $p=\frac{1}{2}(1-\beta)$, $\nu=1+\alpha/2$, $J_\nu(x)$, $I_p(x)$ are the Bessel and modified Bessel functions, respectively, of the first kind, and $A(\lambda)$, $B(\lambda)$ are unknown functions to be found.

The stress component σ_{θ_z} corresponding to this displacement may be obtained by substituting (10) into (2) to get

$$\sigma_{\theta_z}(r,z)=\mu_{\alpha,\beta}r^{\nu-1}z^{1-p}\int_0^\infty \lambda^{1+p}J_\nu(\lambda r)[A(\lambda)I_{p-1}(\lambda z)+B(\lambda)I_{1-p}(\lambda z)]\,\mathrm{d}\lambda. \tag{11}$$

The boundary conditions (5)–(8) reduce to the following two pairs of dual integral equations:

$$\int_0^\infty B(\lambda)J_\nu(\lambda r)\,\mathrm{d}\lambda = 2^{-p}\gamma\Gamma(1-p)r^\nu, \qquad (0\le r\le a), \tag{12}$$

$$\int_0^\infty \lambda^{2p}A(\lambda)J_\nu(\lambda r)\,\mathrm{d}\lambda=0, \qquad (r>a), \tag{13}$$

$$\int_0^\infty \lambda^p J_\nu(\lambda r)[A(\lambda)I_p(\lambda h)+B(\lambda)I_{-p}(\lambda h)]\,\mathrm{d}\lambda = h^{-p}\delta r^\nu, \qquad (0\le r\le b), \tag{14}$$

$$\int_0^\infty \lambda^{1+p}J_\nu(\lambda r)[A(\lambda)I_{p-1}(\lambda h)+B(\lambda)I_{1-p}(\lambda h)\,\mathrm{d}\lambda=0, \qquad (r>b). \tag{15}$$

Setting

$$A(\lambda)I_{p-1}(\lambda h) + B(\lambda)I_{1-p}(\lambda h) = \lambda^{-p}C(\lambda), \tag{16}$$

we may reduce the two pairs of dual integral Equations (12)–(15) to

$$\int_0^\infty A(\lambda)[1+g_1(\lambda h)]J_\nu(\lambda r)\,\mathrm{d}\lambda = -2^{-p}\gamma\Gamma(1-p)r^\nu + \\ + \int_0^\infty \lambda^{-p}C(\lambda)R_1(\lambda h)J_\nu(\lambda r)\,\mathrm{d}\lambda, \quad (0 \le r \le a), \tag{17}$$

$$\int_0^\infty \lambda^{2p}A(\lambda)J_\nu(\lambda r)\,\mathrm{d}\lambda = 0, \quad (r > a), \tag{18}$$

$$\int_0^\infty C(\lambda)[1+g_2(\lambda h)]J_\nu(\lambda r)\,\mathrm{d}\lambda = h^{-p}\delta r^\nu - \\ - \int_0^\infty \lambda^{p}A(\lambda)R_2(\ h)J_\nu(\lambda r)\,\mathrm{d}\lambda, \quad (0 \le r \le b), \tag{19}$$

$$\int_0^\infty \lambda C(\lambda)J_\nu(\lambda r)\,\mathrm{d}\lambda = 0, \quad (r > b), \tag{20}$$

where

$$g_1(\lambda h) = \frac{I_{p-1}(\lambda h)}{I_{1-p}(\lambda h)} - 1, \tag{21}$$

$$g_2(\lambda h) = \frac{I_{-p}(\lambda h)}{I_{1-p}(\lambda h)} - 1, \tag{22}$$

$$R_1(\lambda h) = \frac{1}{I_{1-p}(\lambda h)}, \tag{23}$$

$$R_2(\lambda h) = -\frac{2\sin(\pi p)}{\pi\lambda h}R_1(\lambda h). \tag{24}$$

We solve the last two pairs of dual integral equations by the multiplying-factor method [21]. Applying the operator

$$\frac{2^{1-p}}{\Gamma(p)}x^{\nu-p}\int_x^\infty r^{1-\nu}(r^2-x^2)^{p-1}\,\mathrm{d}r,$$

to Equation (18) and the operator

$$\frac{2^p}{\Gamma(1-p)}x^{p-\nu-1}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x r^{1+\nu}(x^2-r^2)^{-p}\,\mathrm{d}r,$$

to (17) and making use of the integrals ([22], 8.5 (32) & (33))

$$\int_a^\infty x^{1-\nu} J_\nu(xy)(x^2-a^2)^\mu \, dx = 2^\mu \Gamma(\mu+1) a^{1+\mu-\nu} y^{-\mu-1} J_{\nu-\mu-1}(ay),$$
$$y>0,\ \mathrm{Re}\,\mu > -1,\ \mathrm{Re}(\nu-2\mu) > 1/2, \tag{25}$$
$$\int_0^a x^{1+\nu} J_\nu(xy)(a^2-x^2)^\mu \, dx = 2^\mu \Gamma(\mu+1) a^{1+\mu+\nu} y^{-\mu-1} J_{\nu+\mu+1}(ay),$$
$$y>0,\ \mathrm{Re}\,\mu > -1,\ \mathrm{Re}\,\nu > -1, \tag{26}$$

we obtain the following equations:

$$\int_0^\infty \lambda^p A(\lambda)[1+g_1(\lambda h)] J_{\nu-p}(\lambda x) \quad d\lambda = -\frac{\gamma\Gamma(1+\nu)\Gamma(1-p)}{\Gamma(1+\nu-p)} x^{\nu-p} +$$
$$+\int_0^\infty C(\lambda) R_1(\lambda h) J_{\nu-p}(\lambda x)\, d\lambda, \quad (0 \le x < a), \tag{27}$$

$$\int_0^\infty \lambda^p A(\lambda) J_{\nu-p}(\lambda x)\, d\lambda = 0, \qquad (x > a). \tag{28}$$

We set

$$\int_0^\infty \lambda^p A(\lambda) J_{\nu-p}(\lambda x)\, d\lambda = \Phi(x), \qquad (0 \le x < a). \tag{29}$$

Equations (28) and (29) and the Hankel inversion theorem give

$$A(\lambda) = \lambda^{1-p} \int_0^a t\Phi(t) J_{\nu-p}(\lambda t)\, dt. \tag{30}$$

We determine the auxiliary function $\Phi(x)$ from the following Fredholm integral equation of the second kind which is obtained by using (27) and (30):

$$\Phi(x) = -\frac{\gamma\Gamma(\nu+1)\Gamma(1-p)}{\Gamma(\nu-p+1)} x^{\nu-p} - \int_0^a t\Phi(t) L_1(t,x)\, dt +$$
$$+\int_0^\infty C(\lambda) R_1(\lambda h) J_{\nu-p}(\lambda x)\, d\lambda, \qquad (0 \le x < a), \tag{31}$$

where

$$L_1(t,x) = \int_0^\infty \lambda g_1(\lambda h) J_{\nu-p}(\lambda t) J_{\nu-p}(\lambda x)\, d\lambda. \tag{32}$$

Similarly, using the same technique, we find that the solution of the second dual integral Equations (19) and (20) is

$$C(\lambda) = \frac{\lambda^{1/2}}{h^p} \int_0^b t\Psi(t) J_{\nu-1/2}(\lambda t)\, dt, \tag{33}$$

where the auxiliary function $\Psi(t)$ is the solution of the following Fredholm integral equation of the second kind:

$$\begin{aligned} \Psi(x) &= \sqrt{2}\delta \frac{\Gamma(\nu+1)}{\Gamma(\nu+1/2)} x^{\nu-1/2} - \int_0^b t\Psi(t) L_2(t,x)\, dt - \\ &\quad -h^p \int_0^\infty \lambda^{p+1/2} A(\lambda) R_2(\lambda h) J_{\nu-1/2}(\lambda x)\, d\lambda, \qquad (0 \le x < b), \end{aligned} \tag{34}$$

where

$$L_2(t,x) = \int_0^\infty \lambda g_2(\lambda h) J_{\nu-1/2}(\lambda t) J_{\nu-1/2}(\lambda x)\, d\lambda. \tag{35}$$

Using (33) in (31) and (30) in (34), we get the following simultaneous Fredholm integral equations for the two auxiliary functions $\Phi(x)$ and $\Psi(x)$:

$$\begin{aligned} \Phi(x) &= -\frac{\gamma\Gamma(\nu+1)\Gamma(1-p)}{\Gamma(\nu-p+1)} x^{\nu-p} - \int_0^a t\Phi(t) L_1(t,x)\, dt + \\ &\quad + \int_0^b t\Psi(t) M_1(t,x)\, dt, \qquad (0 \le x \le a), \end{aligned} \tag{36}$$

$$\begin{aligned} \Psi(x) &= \sqrt{2}\delta \frac{\Gamma(\nu+1)}{\Gamma(\nu+1/2)} x^{\nu-1/2} - \int_0^b t\Psi(t) L_2(t,x)\, dt - \\ &\quad - \int_0^a t\Phi(t) M_2(t,x)\, dt, \qquad (0 \le x \le b), \end{aligned} \tag{37}$$

where

$$M_1(t,x) = h^{-p} \int_0^\infty \lambda^{1/2} R_1(\lambda h) J_{\nu-1/2}(\lambda t) J_{\nu-p}(\lambda x)\, d\lambda, \tag{38}$$

$$M_2(t,x) = -\frac{2\sin p\pi}{\pi h^{1-2p}} M_1(x,t). \tag{39}$$

In order to reduce Equations (36) and (37) to a convenient form, we set

$$x^{1/2}\Phi(x) = -\frac{\Gamma(\nu+1)\Gamma(1-p)}{\Gamma(\nu-p+1)} \Omega_1(x), \tag{40}$$

$$x^{1/2}\Psi(x) = \frac{\Gamma(\nu+1)\sqrt{2}}{\Gamma(\nu+1/2)} \Omega_2(x). \tag{41}$$

We obtain

$$\begin{aligned}\Omega_1(x) &= \gamma x^{\nu+1/2-p} - \int_0^a \Omega_1(t)G_1(t,x)\,\mathrm{d}t - \\ &\quad - \frac{\Gamma(\nu-p+1)}{h^{p-1/2}\Gamma(\nu+1/2)\Gamma(1-p)}\int_0^b \Omega_2(t)F(t,x)\,\mathrm{d}t, \qquad (0\le x\le a), \end{aligned} \tag{42}$$

$$\begin{aligned}\Omega_2(x) &= \delta x^{\nu} - \int_0^b \Omega_2(t)G_2(t,x)\,\mathrm{d}t - \\ &\quad - \frac{h^{p-1/2}\Gamma(\nu+1/2)}{\Gamma(\nu-p+1)\Gamma(p)}\int_0^a \Omega_1(t)F(x,t)\,\mathrm{d}t, \qquad (0\le x\le b), \end{aligned} \tag{43}$$

where

$$G_1(t,x) = \sqrt{tx}\int_0^\infty \lambda g_1(\lambda h)J_{\nu-p}(\lambda t)J_{\nu-p}(\lambda x)\,\mathrm{d}\lambda, \tag{44}$$

$$G_2(t,x) = \sqrt{tx}\int_0^\infty \lambda g_2(\lambda h)J_{\nu-1/2}(\lambda t)J_{\nu-1/2}(\lambda x)\,\mathrm{d}\lambda, \tag{45}$$

$$F(t,x) = \sqrt{\frac{2tx}{h}}\int_0^\infty \lambda^{1/2}R_1(\lambda h)J_{\nu-1/2}(\lambda t)J_{\nu-p}(\lambda x)\,\mathrm{d}\lambda, \tag{46}$$

4. Expressions for some Physical Quantities

The shear components $\sigma_{\theta_z}(r,0)$ inside the circle $(z=0, 0\le r\le a)$ and $\sigma_{\theta_z}(r,h)$ inside the circle $(z=h, 0\le r\le b)$ are found to be (Appendix)

$$\sigma_{\theta_z}(r,0) = \frac{2\Gamma(\nu+1)}{\Gamma(p)\Gamma(\nu-p+1)}\mu_{\alpha,\beta}r^{2\nu-2}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{t^{1/2+p-\nu}\Omega_1(t)}{(t^2-r^2)^p}\,\mathrm{d}t, \tag{47}$$

$$\sigma_{\theta_z}(r,h) = -\frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+1/2)h^{2p-1}}\mu_{\alpha,\beta}r^{2\nu-2}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^b \frac{t^{1-\nu}\Omega_2(t)}{(t^2-r^2)^{1/2}}\,\mathrm{d}t. \tag{48}$$

The torques applied to the discs at the faces $z=0$, and $z=h$ are, respectively,

$$T_1 = -2\pi\int_0^a r^2\sigma_{\theta_z}(r,0)\,\mathrm{d}r, \tag{49}$$

$$T_2 = -2\pi\int_0^b r^2\sigma_{\theta_z}(r,h)\,\mathrm{d}r. \tag{50}$$

Substituting (47) into (49) and (48) into (50) we finally get the expressions

$$T_1 = 4\pi\mu_{\alpha,\beta}\frac{\Gamma(1-p)}{\Gamma(p)}\left[\frac{\Gamma(\nu+1)}{\Gamma(\nu-p+1)}\right]^2\int_0^a t^{1/2+\nu-p}\Omega_1(t)\,\mathrm{d}t, \tag{51}$$

$$T_2 = -4\pi\mu_{\alpha,\beta}h^{1-2p}\left[\frac{\Gamma(1+\nu)}{\Gamma(\nu+1/2)}\right]^2\int_0^b t^{\nu}\Omega_2(t)\,\mathrm{d}t. \tag{52}$$

The second quantity of interest for applications is the stress intensity factor at the rims of the two discs defined as

$$K_{\mathrm{III}}^{(1)} = \lim_{r\to a-0}(a-r)^p\sigma_{\theta_z}(r,0), \tag{53}$$

$$K_{\mathrm{III}}^{(2)} = \lim_{r\to b-0}(b-r)^{1/2}\sigma_{\theta_z}(r,h). \tag{54}$$

They may be derived from (47) and (48) and are given by

$$K_{\mathrm{III}}^{(1)} = -\frac{2^{1-p}\Gamma(\nu+1)}{\Gamma(p)\Gamma(\nu-p+1)}\mu_{\alpha,\beta}a^{\nu-3/2}\Omega_1(a), \tag{55}$$

$$K_{\mathrm{III}}^{(2)} = \sqrt{\frac{2}{\pi}}\frac{\Gamma(\nu+1)}{\Gamma(\nu+1/2)}\mu_{\alpha,\beta}h^{1-2p}b^{\nu-3/2}\Omega_2(b). \tag{56}$$

5. Special Cases

In this section we derive results for some special cases. The solution for case V is new and is discussed in detail.

Case I

We derive the solution for the problem of torsion of a homogeneous layer by means of two circular discs studied by Singh *et al.* [20] by setting $\alpha = \beta = 0$. The two Fredholm integral Equations (42) and (43) take the form

$$\Omega_1(x) = \gamma x - \int_0^a \Omega_1(t)G(t,x)\,\mathrm{d}t - \frac{1}{\sqrt{\pi}}\int_0^b \Omega_2(t)F(t,x)\,\mathrm{d}t, \qquad (0\le x\le a), \tag{57}$$

$$\Omega_2(x) = \delta x - \int_0^b \Omega_2(t)G(t,x)\,\mathrm{d}t - \frac{1}{\sqrt{\pi}}\int_0^a \Omega_1(t)F(t,x)\,\mathrm{d}t, \qquad (0\le x\le b), \tag{58}$$

where

$$G(t,x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-\lambda h}}{\sinh(\lambda h)}\sin(\lambda t)\sin(\lambda x)\,\mathrm{d}\lambda, \tag{59}$$

$$F(t,x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sinh(\lambda h)}\sin(\lambda t)\sin(\lambda x)\,\mathrm{d}\lambda, \tag{60}$$

which is in agreement with the results of [20].

CASE II

When $b \to 0$, the problem reduces to the Reissner–Sagoci problem for a non-homogeneous large thick plate when the lower surface is stress free [19].

In this case we have $\Omega_2(x) \equiv 0$ and the function $\Omega_1(x)$ satisfies the Fredholm integral equation

$$\Omega_1(x) = \gamma x^{\nu+1/2-p} - \int_0^a \Omega_1(t) G(t,x)\, dt, \quad (0 \le x \le a), \tag{61}$$

with the kernel

$$G(t,x) = \sqrt{tx} \int_0^\infty \lambda \left\{ \frac{I_{p-1}(\lambda h)}{I_{1-p}(\lambda h)} - 1 \right\} J_{\nu-p}(\lambda t) J_{\nu-p}(\lambda x)\, d\lambda. \tag{62}$$

CASE III

When $b \to \infty$ and $\delta = 0$, one obtains the Reissner–Sagoci problem for a non-homogeneous large thick plate when the lower surface is rigidly clamped [19].

In this case we have $\Omega_2(x) \equiv 0$ and the function $\Omega_1(x)$ satisfies the Fredholm integral Equation (61) with the kernel

$$G(t,x) = \sqrt{tx} \int_0^\infty \lambda \left\{ \frac{I_{p}(\lambda h)}{I_{-p}(\lambda h)} - 1 \right\} J_{\nu-p}(\lambda t) J_{\nu-p}(\lambda x)\, d\lambda. \tag{63}$$

CASE IV

For very large values of the thickness h of the plate, the problem reduces to that of the Reissner–Sagoci for a non-homogeneous half-space [15]. In this case, the kernels G_1, G_2 and F tend to zero and we get the exact solution

$$\Omega_1^\infty = \gamma^{\nu+1/2-p}, \qquad \Omega_2^\infty = \delta x^\nu.$$

The torque T_1 and the stress intensity factor $K_{\mathrm{III}}^{(1)}$ take the values

$$\begin{aligned} T^\infty &= 4\pi\gamma\mu_{\alpha,\beta} \frac{\Gamma(1-p)}{\Gamma(p)} \left[\frac{\Gamma(1+\nu)}{\Gamma(\nu-p+1)} \right]^2 \frac{a^{2(\nu+1-p)}}{2(\nu+1-p)}, \\ K^\infty &= -\gamma\mu_{\alpha,\beta} \frac{2^{1-p}\Gamma(\nu+1)}{\Gamma(p)\Gamma(\nu-p+1)} a^{2\nu-1-p}. \end{aligned}$$

CASE V

When $\beta = 0$ (*i.e.* $p = 1/2$), one obtains the solution of the problem when the shear modulus of the plate depends on the radial coordinate only. For this case, we have

$$g_1(\lambda h) = g_2(\lambda h) = \frac{e^{-\lambda h}}{\sinh(\lambda h)}, \qquad R_1(\lambda h) = \sqrt{\frac{\pi\lambda h}{2}}\frac{1}{\sinh(\lambda h)}.$$

The Integral Equations (42) and (43) take the forms:

$$\Omega_1(x) = \gamma x^\nu - \int_0^a \Omega_1(t)G(t,x)\,\mathrm{d}t - \frac{1}{\sqrt{\pi}}\int_0^b \Omega_2(t)F(t,x)\,\mathrm{d}t, \qquad (0 \le x \le a), \quad (64)$$

$$\Omega_2(x) = \delta x^\nu - \int_0^b \Omega_2(t)G(t,x)\,\mathrm{d}t - \frac{1}{\sqrt{\pi}}\int_0^a \Omega_1(t)F(t,x)\,\mathrm{d}t, \qquad (0 \le x \le b), \quad (65)$$

where

$$G(t,x) = \sqrt{tx}\int_0^\infty \frac{\lambda e^{-\lambda h}}{\sinh\lambda h} J_{\nu-1/2}(\lambda t)J_{\nu-1/2}(\lambda x)\,\mathrm{d}\lambda, \quad (66)$$

$$F(t,x) = \sqrt{\pi tx}\int_0^\infty \frac{\lambda}{\sinh\lambda h} J_{\nu-1/2}(\lambda t)J_{\nu-1/2}(\lambda x)\,\mathrm{d}\lambda. \quad (67)$$

6. Numerical Results and Discussion

Numerical techniques [23] may be applied to solve the simultaneous Fredholm Integral Equations (42) and (43) for the auxiliary functions $\Omega_1(x)$ and $\Omega_2(x)$.

For the special Case V, we solve Equations (64) and (65) numerically by replacing them by a finite system of linear algebraic equations, for the general values of the parameters a, b, h, γ, δ and α. These equations were reduced to a suitable number of linear algebraic equations depending on the intervals of integration occurring in their expressions. During the numerical procedure, the kernels (66) and (67), involving an infinite interval of integration were evaluated using a Chebyshev–Laguerre quadrature method with 16 nodes [24]. To illustrate the variation of the torque T_1 necessary to produce the given rotation, Equation (51) was used for different values of the geometrical and inhomogeneity parameters. We made the results nondimensional by dividing by the corresponding value of the torque for the nonhomogeneous half-space, i.e. T^∞. The results for $M = T_1/T^\infty$ were plotted against the relative thickness h/a, as shown in Figures 2–9. As expected, the ratio $M \to 1$ for large values of h. This fact does not seem to hold for the results produced in ref. [19]. We have also calculated the nondimensional stress intensity factor $K = K_{\mathrm{III}}^{(1)}/K^\infty$. The resulting curves were similar in shape to those of M.

As shown in the figures, the quantities of interest we have calculated may be approximated by their corresponding values for the half-space for large inhomogeneity factor α, the range of thickness for which this approximation holds becoming larger (reaching towards unity) as the inhomogeneity factor grows larger.

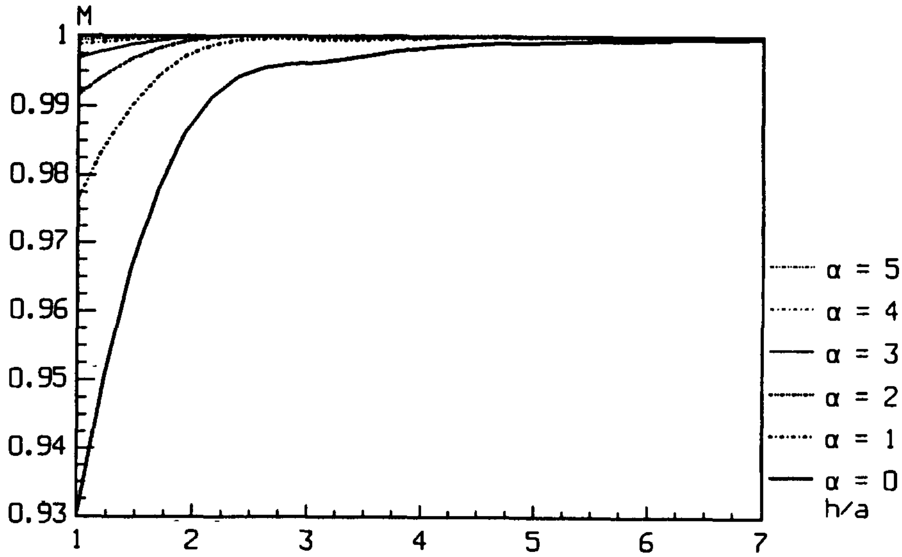

Figure 2. Variation of M for $b = 0$ and $\delta = 0$.

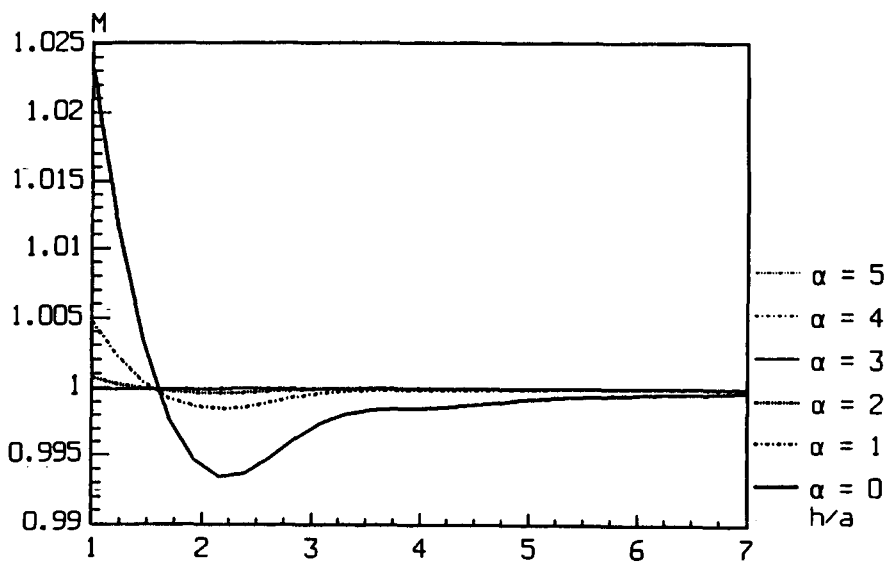

Figure 3. Variation of M for $b = a, \gamma = 0.01$ and $\delta = 0$.

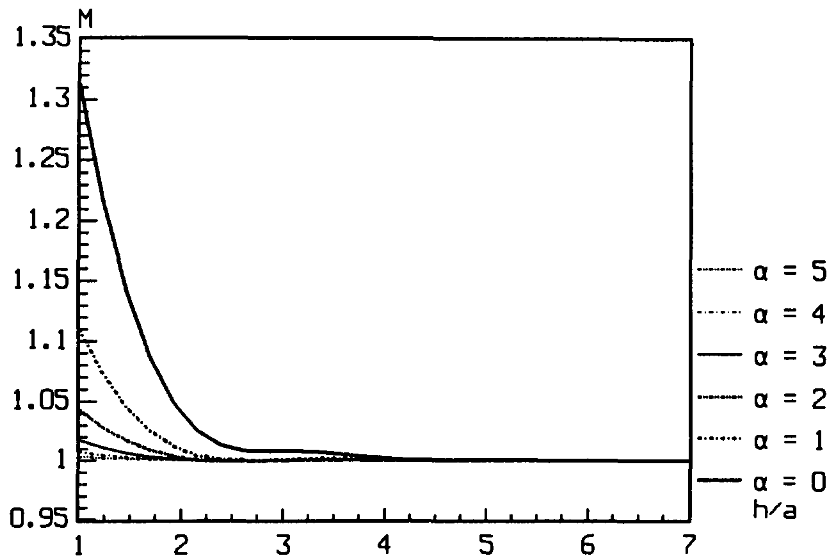

Figure 4. Variation of M for $b = 3a$, $\gamma = 0.01$ and $\delta = 0$.

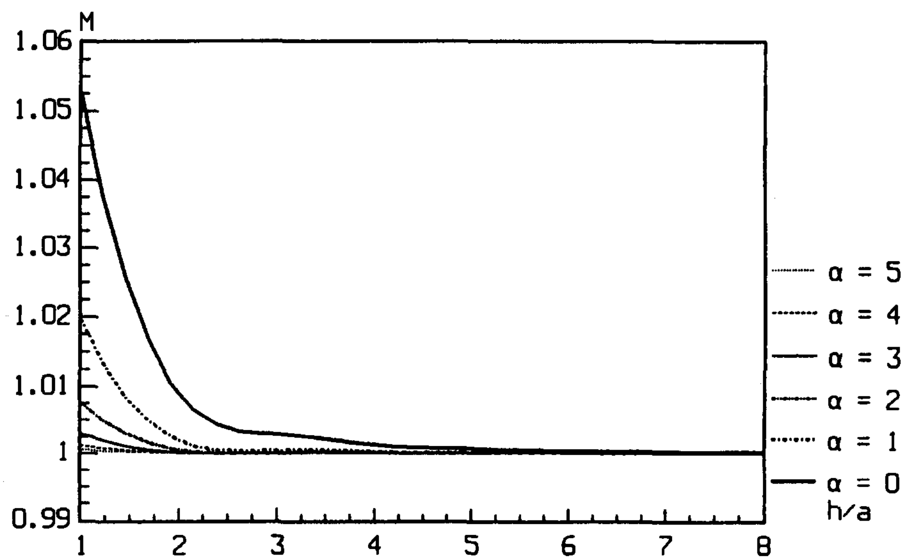

Figure 5. Variation of M for $b \to \infty$, $\gamma = 0.01$ and $\delta = 0$.

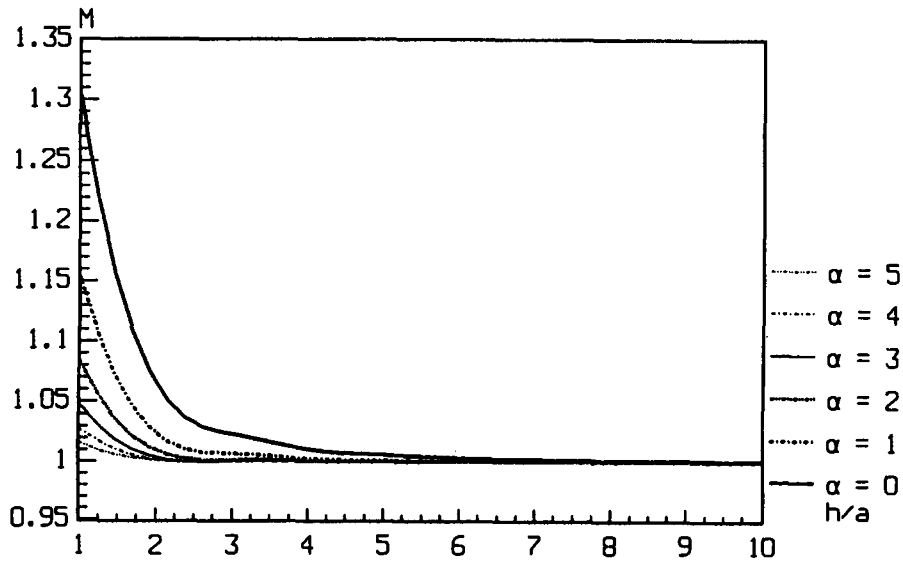

Figure 6. Variation of M for $b = a$, $\gamma = 0.01$ and $\delta = -0.01$.

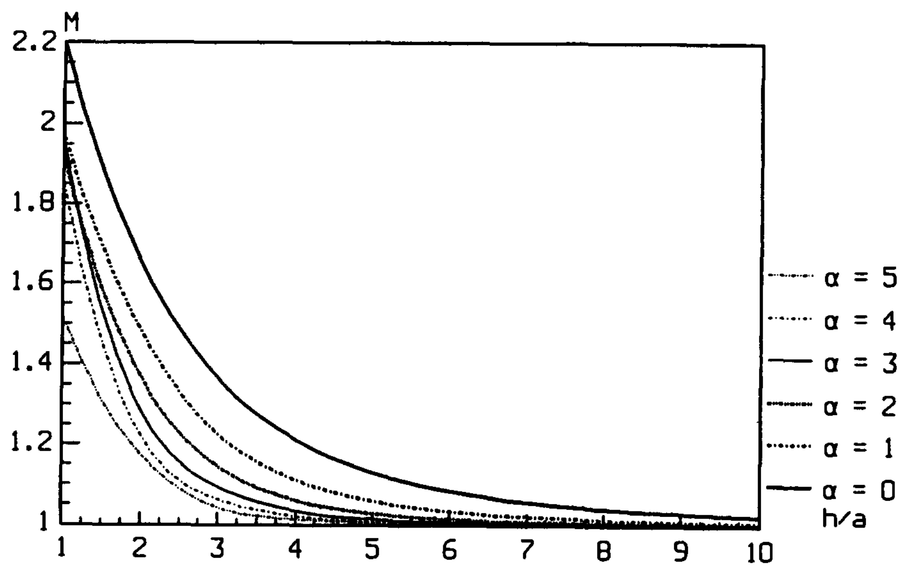

Figure 7. Variation of M for $b = 3a$, $\gamma = 0.01$ and $\delta = -0.01$.

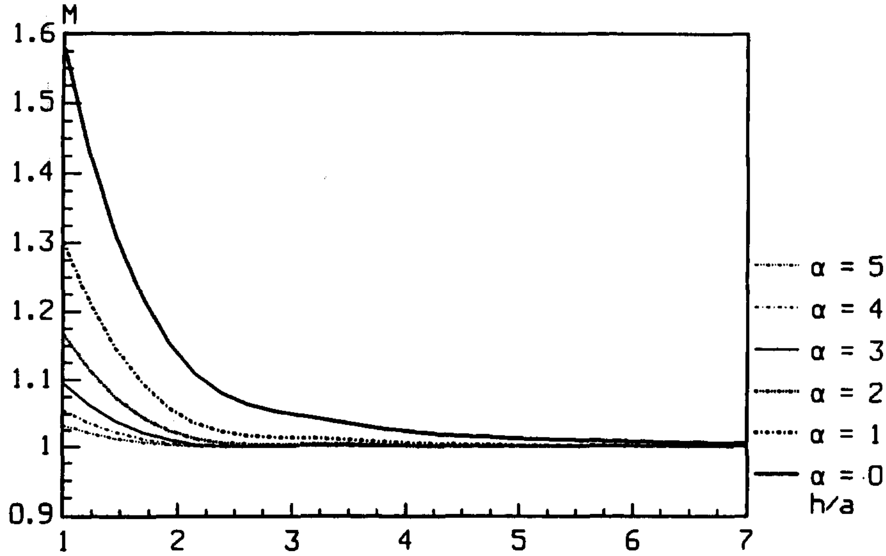

Figure 8. Variation of M for $b = a$, $\gamma = 0.01$ and $\delta = -0.02$.

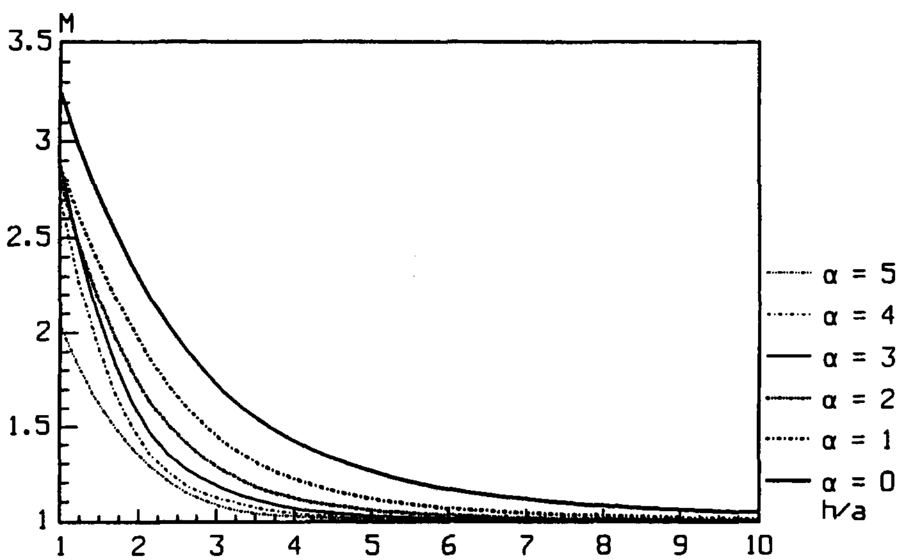

Figure 9. Variation of M for $b = 3a$, $\gamma = 0.01$ and $\delta = -0.02$.

Appendix

Substituting $z = 0$ in (11) and using the results

$$\lim_{x\to 0} x^a I_a(\lambda x) = 0, \quad \lim_{x\to 0} x^a I_{-a}(\lambda x) = \frac{2^a}{\lambda^a \Gamma(1-a)}, \quad (a > 0),$$

we get the following expression for the shear component $\sigma_{\theta_z}(r, 0)$:

$$\sigma_{\theta_z}(r,0) = \frac{2^{1-p}}{\Gamma(p)} \mu_{\alpha,\beta} r^{\nu-1} \int_0^\infty \lambda^{2p} A(\lambda) J_\nu(\lambda r)\,\mathrm{d}\lambda. \tag{68}$$

Substituting for $A(\lambda)$ from (30) into (68) and then interchanging the order of integration, we get

$$\sigma_{\theta_z}(r,0) = \frac{2^{1-p}}{\Gamma(p)} \mu_{\alpha,\beta} r^{\nu-1} \int_0^a t\Phi(t) \left\{ \int_0^\infty \lambda^{1+p} J_\nu(\lambda r) J_{\nu-p}(\lambda t)\,\mathrm{d}\lambda \right\} dt. \tag{69}$$

Making use of the recurrence relation of the Bessel function

$$J_m(\lambda r) = -\frac{1}{\lambda} r^{m-1} \frac{\mathrm{d}}{\mathrm{d}r} [r^{1-m} J_{m-1}(\lambda r)],$$

and the integral ([22], 8.11 (7))

$$\int_0^\infty \lambda^{k-\mu+1} J_\mu(\lambda t) J_k(\lambda r)\,\mathrm{d}\lambda = \frac{2^{k-\mu+1} r^k}{t^\mu \Gamma(\mu-k)(t^2-r^2)^{k-\mu+1}} H(t-r), \quad \mu > k,$$

we observe that the shear component $\sigma_{\theta_z}(r, 0)$ inside the circle $(z = 0, 0 \le r \le a)$ takes the form

$$\sigma_{\theta_z}(r,0) = \frac{-2}{\Gamma(p)\Gamma(1-p)} \mu_{\alpha,\beta} r^{2\nu-2} \frac{\mathrm{d}}{\mathrm{d}r} \int_r^a \frac{t^{1+p-\nu}\Phi(t)}{(t^2-r^2)^p}\,\mathrm{d}t. \tag{70}$$

Also, the shear component $\sigma_{\theta_z}(r, h)$ inside the circle $(z = h, 0 \le r \le b)$ is found from (11) and (16) as

$$\sigma_{\theta_z}(r,h) = \mu_{\alpha,\beta} r^{\nu-1} h^{1-p} \int_0^\infty \lambda C(\lambda) J_\nu(\lambda r)\,\mathrm{d}\lambda. \tag{71}$$

Substituting for $C(\lambda)$ from (33) in (71) and interchanging the order of integration, we obtain

$$\sigma_{\theta_z}(r,h) = \mu_{\alpha,\beta} r^{\nu-1} h^{1-2p} \int_0^b t\Psi(t) \left\{ \int_0^\infty \lambda^{3/2} J_\nu(\lambda r) J_{\nu-1/2}(\lambda t)\,\mathrm{d}\lambda \right\} dt. \tag{72}$$

Using the same technique as in Equation (69), we see that Equation (72) leads to

$$\sigma_{\theta_z}(r,h) = -\sqrt{\frac{2}{\pi}}\, h^{1-2p} \mu_{\alpha,\beta} r^{2\nu-2} \frac{\mathrm{d}}{\mathrm{d}r} \int_r^b \frac{t^{3/2-\nu}\Psi(t)}{\sqrt{t^2-r^2}}\,\mathrm{d}t. \tag{73}$$

Acknowledgements

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